MTH 304: Metric Spaces and Topology
Semester 2, 2016-17

Contents

1 Topological spaces

1.1 Basic definitions and examples

(i) Let X be a set, and $\mathcal{T}$ a collection of subsets of X such that

(a) $\emptyset, X \in \mathcal{T}$,

(b) for an arbitrary index set J, and a subcollection $\{V_\alpha\}_{\alpha \in J} \subset \mathcal{T}$, we have

$$\bigcup_{\alpha \in J} V_\alpha \in \mathcal{T}$$

(i.e $\mathcal{T}$ is closed under arbitrary union), and

(c) for an finite subcollection $\{W_1, \ldots W_n\} \subset \mathcal{T}$, we have

$$\bigcap_{i=1}^{n} W_i \in \mathcal{T}\,.$$

Then $\mathcal{T}$ is said to define a *topology* on X, and the pair $(X, \mathcal{T})$ is called a *topological space.*

(ii) Let $(X, \mathcal{T})$ be a topological space. Then each $U \in \mathcal{T}$ is called an *open* set.

(iii) Let X be a set, and $\mathcal{T}, \mathcal{T}'$ be topologies on X. Then $\mathcal{T}$ is a said to *coarser* (or equivalently, $\mathcal{T}'$ is said to be *finer* than $\mathcal{T}$) if $\mathcal{T} \subset \mathcal{T}'$.

(iv) Examples of topological spaces. Let X be any set.

(a) The collection $\{\emptyset, X\}$ defines a topology on X called the *indiscrete topology* on X. This is the coarsest possible topology that can be defined on X.

(b) The power set $\mathcal{P}(X)$ of X defines a topology on X called the *discrete topology* on X. $\mathcal{P}(X)$ is the finest possible topology that can be defined on X.

(c) The collection $\{S \subset X \mid |X \setminus S| < \infty\}$ defines topology on X called the *cofinite topology* on X.

(d) The collection $\{S \subset X \mid X \setminus S \text{ is countable}\}$ defines a topology on X called the *cocountable topology* on X.

1.2 Basis and subbasis for a topology

(i) Given a set X, and a collection $\mathscr{B}$ of subsets of X such that

(a) for each $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B$, and

(b) for any pair of elements $B_1, B_2 \in \mathscr{B}$ and $y \in B_1 \cap B_2$, there exists $B' \in \mathscr{B}$ such that $y \in B' \subset B_1 \cap B_2$.

Then $\mathscr{B}$ is said to form a *basis for a topology* $\mathcal{T}$ *on* X *(or* $\mathscr{B}$ *generates a topology* $\mathcal{T}$ *on* X) if for each $U \in \mathcal{T}$ and for each $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B \subset U$.

(ii) Let $(X, \mathcal{T})$ be a topological space. Let $\mathscr{B} \subset \mathcal{T}$ be any subccollection such that for each $U \in \mathcal{T}$ and each $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B \in U$. Then $\mathscr{B}$ generates the topology $\mathcal{T}$.

(iii) Let X be a set, and $\mathscr{B}$ a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ is the collection of all arbitrary unions of elements in $\mathscr{B}$.

(iv) Let $\mathscr{B}$ and B' be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively, on X. Then $\mathcal{T} \subset \mathcal{T}'$ if and only if for each $B \in \mathscr{B}$ and each $x \in B$, there exists $B' \in \mathscr{B}'$ such that $x \in \mathscr{B}' \in B$.

(v) Examples of bases for topologies on $X = \mathbb{R}$.

(a) The collection of all open intervals

$$\mathscr{B} = \{(a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\}$$

forms a basis for a topology on $\mathbb{R}$ called the *standard topology.* Note that the real line with this topology is simply denoted by $\mathbb{R}$.

(b) The collection

$$\mathscr{B}_\ell = \{[a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\}$$

is a basis a for a topology $\mathbb{R}_\ell$ on $\mathbb{R}$ called the *lower limit topology.*

(c) The collection

$$\mathscr{B}_K = \{(a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\} \cup \{(a, b) \setminus K \mid a, b \in \mathbb{R} \text{ and } a < b\},$$

where $K = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, is a basis a for a topology $\mathbb{R}_K$ on $\mathbb{R}$ called the *K-topology.* The topologies $\mathbb{R}_K$ and $\mathbb{R}_\ell$ are strictly finer than standard topology $\mathbb{R}$, but they are not comparable.

(vi) Let X be set, and let $\mathcal{S}$ be any collection of subsets of X whose union equals X. Then the collection $\mathscr{B}$ of all finite intersections of elements in S forms a basis for a topology $\mathcal{T}$ on X. Any such collection $\mathcal{S}$ is called a *subbasis* for the topology $\mathcal{T}$, or in other words, $\mathcal{S}$ *generates* $\mathcal{T}$.

1.3 The Product Topology $X \times Y$

(i) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then the collection

$$\mathscr{B} = \{U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}$$

forms a basis for a topology on $X \times Y$ called the *product topology.* The product topology is simply denoted by $X \times Y$.

(ii) Let $\mathscr{B}_X$ be a basis for $(X, \mathcal{T}_X)$ and $\mathscr{B}_y$ a basis for $(Y, \mathcal{T}_Y)$. Then the collection

$$\mathscr{B}_{X \times Y} = \{B \times C \mid B \in \mathscr{B}_X \text{ and } C \in \mathscr{B}_Y\}$$

forms a basis for the product topology $X \times Y$.

(iii) Examples for product topology on $\mathbb{R}^n$.

(a) The collection

$$\{(a, b) \times (c, d)\}$$

of all open rectangles in $\mathbb{R}^2$ (i.e. products of open intervals in $\mathbb{R}$) forms a basis for a topology on $\mathbb{R}^2$ called the *standard topology on* $\mathbb{R}^2$.

(b) Generalizing Example (i), the collection

$$\{\prod_{i=1}^{n}(a_i, b_i)\}$$

of all open n-dimensional cubes in $\mathbb{R}^n$ (i.e. products of n open intervals in $\mathbb{R}$) forms a basis for a topology on $\mathbb{R}^n$ called the *standard topology on* $\mathbb{R}^n$ (denoted simply by $\mathbb{R}^n$).

(c) For $x \in \mathbb{R}^n$ and $r > 0$, consider the open ball

$$B_{x,r} = \{y \in \mathbb{R}^n \mid \|y - x\| < r\}.$$

Then the collection

$$\{B_{x,r} \mid x \in \mathbb{R}^n \text{ and } r > 0\}$$

also forms a basis for the standard topology $\mathbb{R}^n$. In the subject of Real Analysis, this is most preferred topology on $\mathbb{R}^n$.

(iv) For $1 \leq i \leq n$, let $(X_i, \mathcal{T}_i)$ be topological spaces, and let

$$p_i : \prod_{j=1}^{n} X_j \to X_i$$

be the projection map onto the i^{th} coordinate. Then the collection

$$\mathcal{S} = \{p_i^{-1}(U_i) \mid 1 \leq i \leq n \text{ and } U_i \in \mathcal{T}_i\}$$

forms a subbasis for the product topology $\prod_{i=1}^{n} X_i$.

1.4 Subspace topology

(i) Let $(X, \mathcal{T})$ be a topological space, and $Y \subset X$. The collection

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$$

forms a topology on Y called the *subspace topology or the topology inherited by Y from x.* Under this topology, Y is called a *subspace* of X.

(ii) Let $(X, \mathcal{T})$ be a topological space generated by a basis $\mathscr{B}$. Then the collection

$$\mathscr{B}_Y = \{B \cap Y \mid B \in \mathscr{B}\}$$

is a basis for $\mathcal{T}_Y$.

(iii) Let Y be a subspace of $(X, \mathcal{T})$. If $V \in \mathcal{T}_y$ and $Y \in \mathcal{T}$, then $V \in \mathcal{T}$.

(iv) Examples of subspaces.

(a) Let $X = \mathbb{R}$ and $Y = [0, 1]$. Then every basic open set in the subspace topology inherited by $[0, 1]$ from $\mathbb{R}$ has one the following four types:

$$(a, b) \cap Y = \begin{cases} (a, b), & \text{if } a, b \in Y, \\ [0, b), & \text{if only } b \in Y, \\ (a, 1], & \text{if only } a \in Y, \\ Y, \emptyset & \text{if both } a, b \notin Y. \end{cases}$$

(b) The set $\mathbb{Z}$ of integers, and the set $\mathbb{N}$ of natural numbers both inherit the discrete topology from $\mathbb{R}$.

1.5 Closed sets and limit points

(i) Let $(X, \mathcal{T})$ be a topological space. Then $A \subset X$ is called a *closed* set if $X \setminus A \in \mathcal{T}$.

(ii) Examples of closed sets.

(a) In any topological space $(X, \mathcal{T})$, $\emptyset$ and X are closed sets.

(b) The closed interval $[a, b]$ is a closed set in $\mathbb{R}$. In general, the product of n closed intervals

$$\prod_{i=1}^{n} [a_i, b_i]$$

is closed in $\mathbb{R}^n$.

(c) In the subspace $Y = [0, 1] \cup (2, 3)$ of $\mathbb{R}$, the subsets $[0, 1]$ and $(2, 3)$ are both open and closed.

(d) In the cofinite topology, all finite sets are closed.

(e) All sets are closed in the discrete topology.

(iii) Arbitrary intersections and finite unions of closed sets in a topological space are also closed.

(iv) Let Y be a subspace of a topological space $(X, \mathcal{T})$. Then A is closed in Y if, and only if $A = C \cap X$ for some C closed in X.

(v) The $(X, \mathcal{T})$ be a topological space, and $A \subset X$.

(a) The *interior of* A (denoted by A°) is the union of all open sets contained in A, that is,

$$A^\circ = \cup\{V \in \mathcal{T} \mid V \subset A\}$$

(b) The *closure of* A (denoted by $\bar{A}$) is the intersection of all closed sets containing A, that is,

$$\bar{A} = \cap\{C \mid X \setminus A \in \mathcal{T} \text{ and} A \subset C.\}$$

(c) It follows by definition that

$$A^\circ \subset A \subset \bar{A}.$$

(vi) Let Y be a subspace of a topological space X, and let $A \subset Y$. Let $\bar{A}_Y$ denote the closure of A in Y. Then

$$\bar{A}_Y = \bar{A} \cap Y.$$

(vii) Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. Then $x \in \bar{A}$ if, and only if, for every open set $U \ni x$, we have that

$$U \cap X \neq \emptyset.$$

(viii) Let $(X, \mathcal{T})$ be a topological space. Then $A \subset X$ is dense in X if $\bar{A} = X$.

(ix) Examples of closure of subsets of $X = \mathbb{R}$.

(a) If $A = (a, b]$ or $[a, b)$ or (a, b), then $\overline{Y} = [a, b]$.

(b) If $A = \mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$, then $\bar{A} = \mathbb{R}$. Consequently, both the rationals and the irrationals are dense subsets of $\mathbb{R}$.

(c) If $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, then $\bar{A} = A \cup \{0\}$.

(x) Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. Then $x \in A$ is a *limit point* of A if for every open $U \ni x$, we have

$$U \cap (A \setminus \{x\}) \neq \emptyset.$$

The set of all limit points of A will be denoted by A'.

(xi) Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. Then

$$\bar{A} = A \cup A'.$$

Consequently, A is closed if, and only if $A = \bar{A}$.

1.6 T_1 and Hausdorff (T_2) spaces

(i) Let $(X, \mathcal{T})$ be a topological space. Let distinct points $x, y \in X$ can be *separated by open sets* if there exists $U, V \in \mathcal{T}$ with $U \cap V = \emptyset$ such that $x \in U$ and $y \in V$.

(ii) A topological space $(X, \mathcal{T})$ is said to be *Hausdorff or is said to satisfy the* T_2*-axiom* if every pair x, y of distinct points in X can be separated by open sets.

(iii) A topological space $(X, \mathcal{T})$ is said to be T_1 *or is said to satisfy the* T_1*-axiom* if for every $x \in X$, the set $\{x\}$ is closed in X.

(iv) If a topological space is Hausdorff, then it is T_1.

(v) Every finite set of a Hausdorff space is closed.

(vi) Let $(X, \mathcal{T})$ be a topological space, and let (x_n) a sequence of points in x. Then the sequence is said to *converge to* x (in symbols $x_n \to x$) if every open set $U \ni x$ contains all but finitely many points of the sequence x_n.

(vii) Let $(X, \mathcal{T})$ be a T_1 space, and $A \subset X$. Then $x \in A'$ if, and only if every open set $U \ni x$ has infinitely many points of A.

(viii) In a Hausdorff space X, every sequence converges to at most one point in X.

(ix) Examples of Hausdorff and non-Hausdorff spaces.

  (a) $\mathbb{R}^n$ is a Hausdorff space.

  (b) Any set with the discrete topology is a Hausdorff space, as singletons are closed.

  (c) $\mathbb{R}$ with the cofinite topology is non-Hausdorff.

  (d) The set $X = \{a, b, c\}$ with the topology

$$\mathcal{T} = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b\}\}$$

  is non-Hausdorff, as the points a and c cannot be separated by open sets. Moreover, both a and c are limit points of the set $\{b\}$, which implies that $\overline{\{b\}} = X$ or $\{b\}$ is dense in X.

(x) Subspaces and products of Hausdorff spaces are Hausdorff.

1.7 Continuous functions

(i) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f : X \to Y$ is said to be *continuous* if for every $U \in \mathcal{T}_y$, we have $f^{-1}(U) \in \mathcal{T}_X$.

(ii) Examples (or non-examples) of continuous functions.

(a) For any topological space $(X, \mathcal{T}_X)$, and any $c \in X$, constant map

$$e_c : X \to \{c\}$$

is a continuous map.

(b) Any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

(c) For any topological space $(X, \mathcal{T}_X)$, and $A \subset X$, the inclusion map

$$j_A : A \hookrightarrow X$$

is a continuous map.

(d) Any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

(e) The projection map on a finite product topological spaces

$$\pi_i : \prod_{i=1}^{n} X_i \to X_i$$

is continuous.

(f) The identity map between $i_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}_\ell$ is not continuous.

(iii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a map. Then the following are equivalent.

(a) f is continuous.

(b) For every $A \subset X$, we have $f(\bar{A}) \subset \overline{f(A)}$.

(c) For every closed set C of Y, the set $f^{-1}(C)$ is closed in X.

(d) For every $x \in X$, and each $V \in \mathcal{T}_y$ with $f(x) \in V$, there exists $U \in \mathcal{T}_X$ such that $f(U) \subset V$.

(iv) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a bijective map. If both f and f^{-1} are continuous maps, then f is called a *homeomorphism*. When a space X is homemorphic to a space Y, we write $X \approx Y$.

(v) Examples of homeomorphisms.

(a) For $n \geq 1$, the stereographic projection

$$P_n : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$$

is a homeomorphism.

(b) A vector space isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism.

(vi) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a map. Then f is said to be an *open map* if for every $U \in \mathcal{T}_X$, we have $f(U) \in \mathcal{T}_Y$.

(vii) A bijective map $f : X \to Y$ is a homeomorphism if, and only if f is an open and continuous map.

(viii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ an injective continuous map. We say that f is a *topological imbedding* if $f' : X \xrightarrow{f} f(X)$ is a homeomorphism. If $f : X \to Y$ is an imbedding, we say X *imbeds into* Y and write $X \hookrightarrow Y$.

(ix) Examples (and non-examples) of imbeddings.

(a) For $n < m$, the natural inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$ is an imbedding. More generally, an injective linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ an imbedding.

(b) For $n \geq 1$, there is a natural embedding $S^n \hookrightarrow S^{n+1}$ that maps S^n homeomorphically to the equator of S^{n+1}.

(c) The map $f : [0, 1) \to S^1 (\subset \mathbb{C})$ defined by $f(s) = e^{i2\pi s}$ is injective and continuous, but f^{-1} is not continuous. Hence, this is not an imbedding.

(x) The composition of two continuous functions is continuous.

(xi) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ a map.

(a) If f is continuous, then for every $A \subset X$, the restriction $f|_A : A \to Y$ is continuous.

(b) If f is continuous, then the map $f' : X \xrightarrow{f} f(X)$ is continuous.

(c) If $X = \cup_\alpha V_\alpha$, where $V_\alpha \in \mathcal{T}_x$ and $f|_{V_\alpha}$ is continuous for each α, then f is continuous.

(xii) (Pasting Lemma) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let A, B are closed subsets of X such that $X = A \cup B$. Suppose that $f : A \to Y$ and $g : B \to Y$ are continuous maps such that $f|_{A\cap B} = g|_{A\cap B}$. Then f and g can be extended to a continuous map $F : X \to Y$ with $F|_A = f$ and $F|_B = g$.

1.8 Product and box topologies for arbitrary products

(i) Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in J}$ be an indexed family of topological spaces, and let $X = \prod_{\alpha\in J} X_\alpha$.

(a) The collection

$$\mathscr{B}_{box} = \{\prod_{\alpha\in J} U_\alpha \,|\, U_\alpha \in \mathcal{T}_\alpha\}$$

forms a basis for a topology on X called the *box topology*.

(b) The collection $\mathcal{S} = \cup_{\alpha\in J} \mathcal{S}_\alpha$, where

$$\mathcal{S}_\alpha = \{\pi_\alpha^{-1}(U_\alpha) \,|\, U_\alpha \in \mathcal{T}_\alpha\}$$

forms a subbasis for a topology on X called the *product topology*. Consequently, the collection

$$\mathscr{B} = \{\prod_{\alpha\in J} U_\alpha \,|\, U_\alpha \in \mathcal{T}_\alpha \text{ and } U_\alpha = X_\alpha, \text{ for all but finitely many } \alpha\}$$

forms a basis for the product topology on X.

(ii) The product topology is the preferred topology on an arbitrary product of spaces $X = \prod_{\alpha\in J} X_\alpha$, unless otherwise mentioned.

(iii) On a finite product of topological spaces, the product and box topologies will coincide.

(iv) Let $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in J}$ be an indexed family of topological spaces, and let $X = \prod_{\alpha\in J} X_\alpha$. Suppose that $\mathscr{B}_\alpha$ is a basis for $\mathcal{T}_\alpha$. Then

(a) The collection

$$\mathscr{B}_{box} = \{\prod_{\alpha \in J} B_\alpha \mid B_\alpha \in \mathscr{B}_\alpha\}$$

forms a basis for a the box topology on X.

(b) The collection

$$\mathscr{B} = \{\prod_{\alpha \in J} B_\alpha \mid B_\alpha \in \mathscr{B}_\alpha \text{ for finitely many indices and } B_\alpha = X_\alpha, \text{otherwise}\}$$

forms a basis for the product topology on X.

(v) An arbitrary product of Hausdorff topological spaces is Hausdorff in both the product and box topologies.

(vi) Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces, and let $A_\alpha \subset X_\alpha$. Then

$$\prod_{\alpha \in J} \bar{A}_\alpha = \overline{\prod_{\alpha \in J} A_\alpha}.$$

(vii) Let $X = \prod_\alpha X_\alpha$ be given the product topology, where $\{X_\alpha\}_{\alpha \in J}$ is an indexed family of topological spaces, and let Y be any topological space. Then a map $f : Y \to X$ is continuous if, and only if each $f_\alpha = \pi_\alpha \circ f$ is continuous.

(viii) Assertion (vii) above does not hold if X is given the box topology. As a counterexample, consider $\mathbb{R}^\infty$, the product of countably many copies of $\mathbb{R}$, with the box topology, and the map

$$f : \mathbb{R} \to \mathbb{R}^\infty \ : \ t \mapsto (t, t, \ldots), \forall t \in \mathbb{R}.$$

Clearly for each i, $f_i = \pi_i \circ f = i_{\mathbb{R}}$, and hence f_i is continuous. But f is not continuous, as the inverse (under f) of the basic open set $\prod_{i=1}^\infty (-\frac{1}{i}, \frac{1}{i})$ is not open in $\mathbb{R}$.

1.9 Metric topology

(i) A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties:

(a) $d(x, x) \geq 0$, for all $x, y, \in X$, and $d(x, y) = 0 \iff x = y$.

(b) (Symmetry) $d(x, y) = d(y, x)$, for all $x, y \in X$.

(c) (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

If d defines a metric on a set X, then the pair (X, d) is called a *metric space.*

(ii) Let (X, d) be a metric space. For some $r > 0$, the set

$$B_d(x, r) = \{y \in X \mid d(x, y) < r\}$$

is called *open ball centered at x and radius r.*

(iii) Let (X, d) be a metric space. Then the collection

$$\mathscr{B}_d = \{B_d(x, r) \mid x \in X \text{ and } r \in (0, \infty)\}$$

forms a basis for a topology $\mathcal{T}_d$ on X called the *metric topology induced by d (or the topology induced by the metric d).*

(iv) A topological space $(X, \mathcal{T})$ is said to be *metrizable* if there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$.

(v) Example of metrics.

(a) On any set X, for a fixed $r > 0$, the function $b_r : X \times X \to \mathbb{R}$ defined by

$$b_r(x, y) = \begin{cases} r, & \text{if } x \neq y, \text{ and} \\ 0, & \text{if } x = y. \end{cases}$$

defines a metric on X called the *bounded metric*. Note that for any $x \in X$,

$$B_{b_r}(x, k) = \begin{cases} X, & \text{if } k \geq r, \text{ and} \\ \{x\}, & \text{if } k < r. \end{cases}$$

Consequently, $\mathcal{T}_{b_r} = \mathcal{P}(X)$.

(b) The *standard euclidean metric* d on $\mathbb{R}^n$ is defined by

$$d(x, y) = \|x - y\|_2 = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2},$$

for any two vectors $x, y \in \mathbb{R}^n$ with components x_i and y_i, respectively.

(c) The *square metric* ρ on $\mathbb{R}^n$ is defined by

$$\rho(x,y) = \|x-y\|_\infty = \max_{1\le i\le n} |x_i - y_i|,$$

for any two vectors $x, y \in \mathbb{R}^n$ with components x_i and y_i, respectively.

(d) For any metric space (X,d), and some $r>0$, the map $\bar{d}_r : X\times X \to \mathbb{R}$ defined by

$$\bar{d}_r(x,y) = \min(d(x,y), r)$$

also defines a metric on X such that $\mathcal{T}_d = \mathcal{T}_{\bar{d}_r}$.

(vi) Let d and d' be two metrics defined on X. Then $\mathcal{T}_d \subset \mathcal{T}'_d$ if, and only if for each $x\in X$ and each $r>0$, there exists $r'>0$ such that $B_{d'}(x,r') \subset B_d(x,r)$.

(vii) The square metric and the standard euclidean metric both induce the standard (product) topology on $\mathbb{R}^n$.

(viii) Given an index set J, and points $x=(x_\alpha)$ and $y=(y_\alpha)$ in $\mathbb{R}^J$, the map

$$\bar{\rho}(x,y) = \sup\{\bar{d}_1(x_\alpha, y_\alpha) = \min(|x_\alpha - y_\alpha|, 1) \mid \alpha \in J\}$$

defines a metric on $\mathbb{R}^J$ called the *uniform metric.*

(ix) The uniform metric is finer than the product topology on $\mathbb{R}^J$, but coarser than the box topology on $\mathbb{R}^J$. These topologies are different when J is infinite.

(x) For any two points $x=(x_i)$ and $y=(y_i)$ in $\mathbb{R}^\infty$, the function

$$D(x,y) = \sup\left\{\frac{\bar{d}_1(x,y)}{i} \mid i \in \mathbb{N}\right\}$$

defines a metric on $\mathbb{R}^\infty$ that induces the product topology on $\mathbb{R}^\infty$.

(xi) (Sequence Lemma) Let X be a topological space, and let $A\subset X$. If (x_n) be a sequence of points in A such that $x_n \to x$, then $x\in\bar{A}$. The converse holds if X is metrizable.

(xii) Let X and Y be topological spaces, and let $f : X \to Y$ be a map. If f is continuous, then for every sequence (x_n) of points in X such that $x_n \to x$, we have that $f(x_n) \to f(x)$. The converse holds if X is metrizable.

(xiii) Let X be a topological space, and let $f, g : X \to \mathbb{R}$ be continuous maps. Then $f \pm g$, fg, and f/g (defined on all points such that $g(x) \neq 0$) are continuous maps.

(xiv) Let $f_n : X \to Y$ be a sequence of functions from a set X to a metric space (Y, d). Then (f_n) *converges to f: $X \to Y$ uniformly* if given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in X$, $d(f_n(x), f(x)) < \epsilon$, whenever $n > N$.

(xv) (Uniform limit theorem) Let $f_n : X \to Y$ be a sequence of functions from a topological space X to a metric space (Y, d). If $f_n \to f$ uniformly, then f is continuous.

(xvi) $\mathbb{R}^\infty$ with the box topology is not metrizable. Consider its subset

$$A = \{(x_1, x_2, \ldots) \,|\, x_i > 0, \forall i \in \mathbb{N}\}.$$

Note that $0 \in \bar{A}$, but there is no sequence of points in A coverging to 0. Hence, $\mathbb{R}^\infty$ violates the sequence lemma, which makes it non-metrizable.

1.10 Quotient topology

(i) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $p : X \to Y$ be a surjective map. Then p is said to be a *quotient map* provided that

$$U \in \mathcal{T}_Y \iff p^{-1}(U) \in \mathcal{T}_X\,.$$

(ii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $p : X \to Y$ be a surjective map. Then a subset $C \subset X$ is said to be *saturated with respect to p*, if $C = p^{-1}(D)$, for some $D \subset Y$.

(iii) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $p : X \to Y$ be a surjective map. Then p is a quotient map, if and only if, it maps saturated open sets in X to open sets in Y.

(iv) Any sujective, continuous map $p : X \to Y$ that is either an open or a closed map is a quotient map.

(v) Examples of quotient maps.

(a) A homeomorphism $h : X \to Y$ is a quotient map.

(b) Let X_i, for $1 \leq i \leq n$, be topological spaces. Then the projection map

$$\pi_i : \prod_{j=1}^{n} X_j \to X_i$$

is a surjective, continuous, and open map, and hence a quotient map.

(c) In general, a projection map is not a closed map, for example, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is not a closed map. To see this, consider the set $A = \{(x, y) \in \mathbb{R}^2 \ xy = 1\}$, which is closed in $\mathbb{R}^2$ (by Problem 3, Homework 2), but $f(A) = (\infty, 0) \cup (0, \infty)$, which is open in $\mathbb{R}^2$.

(d) The map $f : [0, 1] \cup [2, 3] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \in [0, 1], \text{ and} \\ x - 1, & \text{if } x \in [2, 3]. \end{cases}$$

is a surjective, continuous, and closed map. But f is not a open map, as $f([0, 1]) = [0, 1]$, which is not open in $\mathbb{R}$.

(vi) Let $(X, \mathcal{T})$ be a topological space, A a set, and $p : X \to A$ be a surjective map. Then there is a unique topology $\mathcal{T}_A$ on A that makes p a quotient map, which is defined by

$$U \in \mathcal{T}_A \iff p^{-1}(U) \in \mathcal{T}.$$

The topology $\mathcal{T}_A$ on A is called the *quotient topology induced by p.*

(vii) Let X be a topological space, and let $\sim$ be an equivalence relation (or equivalently, we can consider a partition) on X. Let $X/\sim$ denote the set of all equivalence classes of X under $\sim$, and consider the natural surjective map

$$p : X \to X/\sim \,:\, x \mapsto [x].$$

Then the quotient topology on $X/\sim$ induced by p is called the *quotient space of X under $\sim$,* and we also denote this space by $X/\sim$.

(viii) Examples of quotient spaces.

(a) Let $p : \mathbb{R} \to X = \{a, b, c\}$ be defined by

$$p(x) = \begin{cases} a, & \text{if } x > 0, \\ b, & \text{if } x < 0, \text{ and} \\ c, & \text{if } x = 0. \end{cases}$$

Then the quotient topology $\mathcal{T}_X$ on X induced by p is

$$\mathcal{T}_X = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$$

(b) Consider the equivalence relation $\sim$ on $\mathbb{R}$ defined by $x \sim y \iff x - y \in \mathbb{Z}$. Note that under $\sim$, each equivalence class is represented by a real number $[0, 1]$, but no pair of points, expect $\{0, 1\}$ are related in $[0, 1]$. Hence, the quotient space on $\mathbb{R}/\sim$ induced by $p : \mathbb{R} \to \mathbb{R}/\sim$ is homeomorphic to $[0, 1]/0 \sim 1 \approx S^1$.

(c) A natural extension of (a) is the equivalence relation $\sim$ on $\mathbb{R}^2$ defined by

$$(a, b) \sim (c, d) \iff (c - a, d - b) \in \mathbb{Z}^2.$$

Extending the argument in (a), we can see that

$$\mathbb{R}^2/\sim \approx S^1 \times S^1.$$

Note that the equivalence classes under this relation are precisely the orbits of the action of the group $\mathbb{Z}^2$ on the group $\mathbb{R}^2$ defined by

$$(m, n) \cdot (x, y) = (x + m, y + n).$$

Consequently, the quotient space $\mathbb{R}^2/\sim$ is often written as $\mathbb{R}^2/\mathbb{Z}^2$.

(d) For $n \geq 1$, the equivalence relation $\sim$ on the open unit n-ball $D^n = B_d(0, 1)$ in $\mathbb{R}^n$ defined by

$$x \sim y \iff \|x\| = \|y\| = 1$$

collapses $\partial D^n = S^{n-1}$ to a single point, and the quotient space thus obtained $\approx S^n$.

(ix) Let $p : X \to Y$ be a quotient map, and let $A \subset X$ be saturated with respect to p; let $q = p|_A$.

  (a) If A is either an open or a closed set, then q is a quotient map.

  (b) If p is either an open or a closed map, then q is a quotient map.

(x) Let p be a quotient map. Let Z be a space, and let $g : X \to Z$ be a map that is constant on each fiber $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. Moreover, f is a continuous if and only if g is continuous, and f is a quotient map if and only if g is a quotient map.

(xi) Let $g : X \to Z$ be a surjective continuous map. Consider the equivalence relation $\sim$ on X defined by

$$x \sim y \iff g(x) = g(y),$$

and let $X^* = X/\sim$.

  (a) Then g induces a bijective continuous map $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.

  (b) If Z is Hausdorff, then so is X^*.

(xii) Some counterexamples.

  (a) Consider the subspaces

$$X = \bigcup_{n=1}^{\infty} [0,1] \times \{n\}$$

and

$$Y = \{(x, x/n) \,|\, x \in [0,1] \text{ and } n \in \mathbb{N}\}$$

of $\mathbb{R}^2$. The map

$$g : X \to Z \,:\, (x,n) \mapsto (x, x/n)$$

is surjective and continuous. Under the equivalence relation $\sim$ on X induced by g (as in assertion (xi) above), the quotient space $X^* = X/\sim$ is obtained by simply collapsing $\{0\} \times \mathbb{N} \subset X$ to a point. Consequently, by Assertion (xi), g induces a bijective continuous map $f : X^* \to Z$. As the image of the closed set

$$A = \{(1/n, n) \mid n \in \mathbb{N}\}$$

under f is not closed, f is not a homeomorphism.

(b) The product of two quotient maps need not be a quotient map. Consider $X = \mathbb{R}_K$ and the quotient space on Y obtained from X by collapsing the subset K to a point. Let $p : \mathbb{R}_K \to Y$ be the induced quotient map. Since $(p \times p)^{-1}(\Delta_Y)$ (Δ_Y denotes the diagonal of Y) is not closed in $\mathbb{R}^2_K$, $p \times p$ is not a quotient map.

1.11 Connected and path connected spaces

(i) Let X be a topological space. A *separation* for X is a pair U, V of disjoint open sets in X such that $X = U \sqcup V$.

(ii) A space X is said to be *connected* if there exists no separation for X.

(iii) A space X is connected if, and only if, the only subsets of X that are both open and closed are X and $\emptyset$.

(iv) A subspace Y of a topological space X is connected if, and only if, there exists no pair of disjoint subsets $A, B \subset Y$ such that

$$A' \cap B = A \cap B' = \emptyset.$$

(v) Let C, D form a separation for a space X, and $Y \subset X$ is a connected subset. Then either $Y \subset C$ or $Y \subset D$.

(vi) Let $\{Y_\alpha\}_{\alpha \in J}$ be an arbitrary collection of connected subspaces of a topological space X such that $\bigcap_{\alpha \in J} Y_\alpha \neq \emptyset$. Then $\bigcup_{\alpha \in J} Y_\alpha$ is connected.

(vii) Let A be a connected subspace of a topological space X. If $A \subset B \subset \bar{A}$, then B is connected. Equivalently, if a $A \subset X$ is connected, then for any subset $S \subset A'$, the set $A \cup S$ is also connected. Consequently, if $A \subset X$ is connected, then $\bar{A}$ is also connected.

(viii) Examples of connected and disconnected spaces.

(a) Any space X with the indiscrete topology $\{\emptyset, X\}$ is connected.

(b) Any space X with $|X| > 2$ with the discrete topology is not connected.

(c) An open interval $(a, b) \subset \mathbb{R}$ is connected, as separation for (a, b) would contradict the fact that it is an open interval (why?). It now follows from assertion (vii) and (viii) that all intervals and open rays of $\mathbb{R}$ (i.e subsets of the form $(-\infty, b)$ and (a, ∞) are also connected.

(d) The subspace $[-1, 0) \sqcup (0, 1] \subset \mathbb{R}$ is not connected.

(e) The subspace $\mathbb{Q} \subset \mathbb{R}$ is not connected.

(ix) The continuous image of a connected space is connected.

(x) A finite cartesian product connected spaces is connected.

(xi) Examples in product spaces.

(a) The n-dimensional euclidean space $\mathbb{R}^n$ is connected.

(b) The space $X = \mathbb{R}^\infty$ in the box topology is not connected. This is because the subset A of bounded sequences in X and the subset $B = X \setminus A$ of unbounded sequences in X form a separation for X.

(c) The space $X = \mathbb{R}^\infty$ in the product topology is connected. For each $n \in \mathbb{N}$, consider the subset

$$Y_n = \{(x_1, x_2, \ldots) \in X \mid x_i = 0, \text{ for } i > n\}$$

of X. Since $Y_n \approx \mathbb{R}^n$, Y_n is connected. Furthermore, since

$$\bigcup_{n=1}^{\infty} Y_n = \mathbb{R}_0^\infty \subset X,$$

and $\bar{\mathbb{R}}_0^\infty = X$, it follows from assertion (viii) above that X is connected.

(xii) (Intermediate value theorem) Let X be a connected space, and $f : X \to \mathbb{R}$ be a continuous map. If a and b are distinct points in X, and r is a point lying between $f(a)$ and $f(b)$, then there exists a $c \in X$ such that $f(c) = r$.

(xiii) Let X be a topological space. A *path in X from a point x to a point y* is a continuous map $f : [0,1] \to \mathbb{R}$ such that $f(0) = x$ and $f(1) = y$.

(xiv) A space X is *path connected* if for every pair of distinct point $x, y \in X$ there exists a path in X from x to y.

(xv) A path-connected space is connected.

(xvi) The continuous image of a path connected space is path connected.

(xvii) Examples (or non-examples) of path-connected spaces.

(a) The open unit ball $B_d(0,1) \subset \mathbb{R}^n$ is path connected. This is because for an two distinct points (or vectors) $x, y \in B_d(0,1)$, the continuous map

$$f : [0,1] \to \mathbb{B}_d(0,1) \; : \; t \mapsto (1-t)x + ty$$

defines a straight line in $B_d(0,1)$ (or a path) connecting x to y. Since $B_d(0,1) \approx \mathbb{R}^n$, $\mathbb{R}^n$ is also path-connected.

(b) Let $Y \subset \mathbb{R}^n$ be a countable subset. Then $\mathbb{R}^n \setminus Y$ is path-connected (and hence connected).

(c) For $n \geq 1$, the unit sphere $S^n \subset \mathbb{R}^{n+1}$ is path connected, and the map

$$\mathbb{R}^{n+1} \setminus \{0\} \to S^n \; : \; x \mapsto x/\|x\|$$

is continuous.

(d) A connected topological spaces need not be path connected. Consider the subset

$$S = \{(x, \sin(1/x)) \,|\, x \in (0,1]\} \subset \mathbb{R}^2.$$

Then the set $\bar{S}$, called the *topologist's sine curve*, is a connected subspace of $\mathbb{R}^2$ that is not path connected. This is because $\{(0,0)\} \times [-1,1] \subset \bar{S}$, but (as discussed in class) there is no path in $\bar{C}$ connecting the origin $(0,0) \in \bar{S}$ to any point $(x,y) \in S (\subset \bar{S})$. The connectedness of $\bar{S}$ follows from (viii) above, and the fact that S is connected (being the continuous image of a connected space).

(xviii) Let X be a topological space. Consider the relation $\sim$ on X defined by

$$x \sim y \iff \exists \text{ a connected subset } C \ni x, y.$$

Then $\sim$ defines is an equivalence relation on X, and the equivalence classes thus obtained are called the *components* of X.

(xix) Let X be a topological space. Consider the relation $\sim$ on X defined by

$$x \sim y \iff \exists \text{ a path in X from } x \text{ to } y.$$

Then $\sim$ defines is an equivalence relation on X, and the equivalence classes thus obtained are called the *path components* of X.

(xx) The components (or path components) of X are connected (or path connected) disjoint subspaces of X whose union equals X. Moreover, each nonempty connected (or path connected) subspace of X lies in one of them.

(xxi) Examples of components in spaces.

(a) The connected (and path) components of $\mathbb{Q} \subset \mathbb{R}$ are the singletons. This is also true for the discrete topology on any space X with $|X| \geq 2$.

(b) The topologist's sine curve $\bar{S}$ has a single connected component (as it's connected), but has exactly two path components, namely

$$\bar{S} = S \cup (\{0\} \times [-1, 1]), \text{ where } \{0\} \times [-1, 1] = S' \setminus S.$$

(xxii) A space X is said to be *locally connected (or locally path connected)* if for each $x \in X$, and each neighborhood U of x, there exists a connected (or path connected) neighborhood C of x such that $C \subset U$.

(xxiii) A locally path connected space is also locally connected. However, the connectedness (or the path connectedness) of a space does not necessarily imply its local connectedness (or local path connectedness), and vice versa.

(xxiv) Examples of local connectedness (or local path connectedness).

(a) The subset $[-1, 0] \sqcup (0, 1] \subset \mathbb{R}$ is locally connected, but not connected.

(b) The topologist's sine curve is connected, but not locally connected.

(c) The rationals are neither connected or locally connected.

(xxv) A space X is locally connected (or locally path connected) if, and only if, for every open set U of X, each component (or path component) of U is open in X.

(xxvi) Each path component of a topological space X is a subset of a component of X. Moreover, if X is locally path connected, then its path components and connected components coincide.

1.12 Compactness

(i) An *open cover* for a topological space is X is a collection of sets $\{V_\alpha\}_{\alpha\in J}$ in X such that

$$X = \bigcup_{\alpha\in J} V_\alpha.$$

An *open cover for a subspace* $A \subset X$ is a collection $\{U_\beta\}_{\beta\in J'}$ of open sets in X such that

$$A \subset \bigcup_{\beta\in J'} U_\beta.$$

(ii) A *finite subcover* of an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha\in J}$ for a topological space X is a subcollection $\{V_{\alpha_1}, \dots, V_{\alpha_n}\} \subset \mathcal{V}$ that covers X.

(iii) A space X is said to be *compact* if every open cover for X has a finite subcover.

(iv) Examples of compact (and noncompact) spaces.

(a) Finite topological spaces are compact.

(b) For $n \geq 1$, the space $X = \mathbb{R}^n$ is not compact. This is because the collection of open balls $\{B_d(0, i) \,|\, i \in \mathbb{N}\}$ is an open cover for X that has no finite subcover.

(c) The subset $A = \{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{0\} \subset \mathbb{R}$ is a compact subspace. This is because 0 is a limit point of the sequence $(1/n)$, and so every neighborhood of 0 contains all but finitely terms of the sequence.

(v) A continuous image of a compact space is compact.

(vi) A closed subspace of a compact space is compact.

(vii) A compact subspace of a Hausdorff space is closed.

(viii) Let X be a compact space, and Y be a Hausdorff space. Then a bijective continuous map $f : X \to Y$ is a homeomorphism.

(ix) (Tube Lemma) Let $X \times Y$ be the product of a topological space X with a compact topological space Y. If N is a neighbourhood that contains the slice $\{x\} \times Y$, for some $x \in X$, then there exists a neighborhood U of x in X such that N contains the tube $U \times Y$ about $\{x\} \times Y$.

(x) A collection $\mathcal{C}$ of subsets of X is said to possess the *finite intersection property* if every finite subcollection $\{C_1, \ldots, C_n\} \subset \mathcal{C}$ satisfies

$$\bigcap_{i=1}^{n} C_i \neq \emptyset.$$

(xi) A space X is compact if, and only if, for every collection $\mathcal{C}$ of closed subsets of X satisfying the finite intersection property, we have

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset.$$

(xii) Let X be a set. Then there exists a collection $\mathscr{D}$ of subsets of X that is maximal with respect to the finite intersection property.

(xiii) Let X be a set, and let $\mathscr{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:

(a) $\mathscr{D}$ is closed under finite intersections, and

(b) if a subset A of X intersects every elements $D \in \mathscr{D}$, then $A \in \mathscr{D}$.

(xiv) (Tychonoff theorem) An arbitrary product of compact spaces is compact.

(xv) Every closed interval in $\mathbb{R}$ is compact. Consequently, closed cubes $\prod_{i=1}^{n}[a_i, b_i]$ and closed balls $\overline{B_d(x, r)}$ are compact subsets of $\mathbb{R}^n$.

(xvi) A subset A of $\mathbb{R}^n$ is compact if, and only if it is closed and bounded under the standard euclidean metric d.

(xvii) (Extreme value theorem) Let X be a compact space. If $f : X \to \mathbb{R}$ is a continuous map, then there exists points $c, d \in X$ such that

$$f(c) \leq f(x) \leq f(d),$$

for all $x \in X$.

(xviii) (Lebesque number lemma) Let $\mathcal{V}$ be an open cover of a compact metric space (X, d). Then there exists $\delta > 0$ such that for every subset $A \subset X$ with $\operatorname{diam}(A) < \delta$, there exists $V \in \mathcal{V}$ such that $A \subset V$.

(xix) (Uniform continuity theorem) Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.

(xx) A point x of a topological space X is said to be *isolated* if $\{x\}$ is an open set of X.

(xxi) Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable. Consequently, every closed interval in $\mathbb{R}$ is uncountable, and so $\mathbb{R}$ is uncountable.

(xxii) A space X is said to be *limit point compact* if every infinite subset of X has a limit point.

(xxiii) A compact space is limit point compact, but the converse does not hold true.

(xxiv) Example: Give a space Y the indiscrete topology. The space $X = \mathbb{N} \times Y$ is limit point compact, as every subset of X has a limit point. But X is not compact, as $\{\{n\} \times Y\}_{n \in \mathbb{N}}$ is open cover for X, which has no finite subcover.

(xxv) A space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence.

(xxvi) Let X be a metrizable space. Then the following statements are equivalent.

(a) X is compact.

(b) X is limit point compact.

(c) X is sequentially compact.

(xxvii) A space X is said to be *locally compact* if every point $x \in X$ has a neighborhood U such that $\bar{U}$ is compact.

(xxviii) Examples of locally compact spaces.

(a) The space $\mathbb{R}^n$ is locally compact.

(b) The space $X = \mathbb{R}^\infty$ in the product topology is not locally compact. This is because the closure of every basic open set of X is of the form $\prod_{i=1}^{\infty} C_i$, where

$$C_i = \begin{cases} [a_i, b_i], & \text{for } i = n_1, \ldots n_k, \\ \mathbb{R}, & \text{otherwise.} \end{cases}$$

(xxix) An open or a closed subset of a locally compact Hausdorff space is locally compact.

(xxx) A space Y is said to be the *one point compactification* of a space X if

(a) Y is a compact Hausdorff space,

(b) $X \subset Y$ with $|Y \setminus X| = 1$, and

(c) $Y = \bar{X}$.

(xxxi) A space X is a locally compact Hausdorff if, and only if X has a one point compactification Y.

(xxxii) Example: For $n \geq 1$, the one-point compactification of $\mathbb{R}^n$ is S^n.

1.13 Countability axioms

(i) A topological space X is said to have a *countable local basis at a point* $x \in X$, if there exists a collection $\{U_n\}_{n \in \mathbb{Z}^+}$ of neighborhoods of x such that any other neighborhood contains at least one of the neighborhoods U_n.

(ii) A topological space X is said to be *first countable* if it has a countable local basis at each point $x \in X$.

(iii) Example: Any metric space (X, d) is first countable, as every point $x \in X$ has a countable local basis given by $\{B_d(x, \frac{1}{n})\}_{n \in \mathbb{Z}^+}$.

(iv) (Sequence Lemma) Let X be a topological space, and let $A \subset X$. If (x_n) be a sequence of points in A such that $x_n \to x$, then $x \in \bar{A}$. The converse holds if X is first countable.

(v) Let X and Y be topological spaces, and let $f : X \to Y$ be a map. If f is continuous, then for every sequence (x_n) of points in X such that $x_n \to x$, we have that $f(x_n) \to f(x)$. The converse holds if X is first countable.

(vi) A topological space X is said to be *second countable* if it has a countable basis.

(vii) A second countable space is also first countable.

(viii) Examples of first and second countable spaces.

(a) The space $X = \mathbb{R}^n$ is second countable, as the countable collection of open cubes

$$\mathscr{B} = \{\prod_{i=1}^{n}(a_i, b_i) \,|\, a_i, b_i \in \mathbb{Q} \text{ and } a_i < b_i\}$$

forms a basis for X.

(b) The space $X = \mathbb{R}^\infty$ is second countable, as the collection

$$\mathscr{B} = \{\prod_{i=1}^{\infty} U_i\}, \text{ where}$$

$$U_i = \begin{cases} (a_i, b_i), \text{ with } a_i, b_i \in \mathbb{Q}, & \text{for finitely many i, and} \\ \mathbb{R}, & \text{otherwise,} \end{cases}$$

forms a countable basis for X.

(c) The space $X = \mathbb{R}^\infty$ with the uniform topology satisfies is first countable, as it is metrizable. However, X is not second countable. This is because the second countability of X would imply that any discrete subspace A of X is countable. But this is clearly untrue, as

$$A = \prod_{i=1}^{\infty} \{0, 1\}$$

is an uncountable discrete subspace of X.

(d) The space $X = \mathbb{R}_\ell$ is first countable, as the collection

$$\mathscr{B}_x = \{[x, x + \frac{1}{n}) \,|\, n \in \mathbb{N}\}$$

for a copuntable local basis at each point $x \in X$. But X is not second countable for the following reason. Suppose we assume on the contrary that X has a countable basis $\mathscr{B}$. Then for each $x \in X$, there exists $B_x \in \mathscr{B}$ such that $x \in B_x \in [x, x+1)$. But this would imply that $\mathscr{B}$ is uncountable, which contradicts our hypothesis.

(ix) Subspaces and countable products of first (or second countable) spaces is first (or second countable).

(x) A space X is said to be *separable* if it has countable dense subset.

(xi) Example: $\mathbb{R}^n$ is separable, as it has a countable and dense subset $\mathbb{Q}^n$.

(xii) A space X is said to be *Lindelöf* if every open cover for X has a countable subcover.

(xiii) A second countable space is both separable and Lindelöf.

(xiv) A product of Lindelöf spaces need not be Lindelöf. For example, $\mathbb{R}_\ell$ is Lindelöf, but $Y = \mathbb{R}_\ell^2$ is not, for consider the subset

$$L = \{(x, -x) \,|, x \in \mathbb{R}_\ell\}.$$

Since L is closed in $\mathbb{R}_\ell$, the collection

$$\{Y \setminus L\} \cup \{[a, b) \times [-a, d) \,|\, a, b, c, d \in \mathbb{R}_\ell\}$$

forms an open cover for Y. If Y has countable subsover, then $\mathbb{R}_\ell$ should be countable, which is impossible.

1.14 Separation axioms

(i) Let $(X, \mathcal{T})$ be a topological space, and $\{A, B\}$ be a pair of disjoint sets in X. The $\{A, B\}$ can be *separated by open sets* if there exists open sets $U, V \in \mathcal{T}$ with $U \cap V = \emptyset$ such that $A \subset U$ and $B \subset V$.

(ii) Let X be a T_1 space. Then:

(a) X is said to be *regular* if each pair of subsets of X of the form $\{\{x\}, B\}$, where B is closed and $x \in B \setminus X$, can separated by open sets.

(b) X is said to be *normal* if each pair of subsets of X of the form $\{A, B\}$, where A and B are disjoint closed sets, can separated by open sets.

(iii) Let X be a T_1 space. Then:

(a) X is regular if, and only if, for each point $x \in X$ and each neighborhood $U \ni x$, there exists a neighborhood $V \ni x$ such that $\bar{V} \subset U$.

(b) X is normal if, and only if, for each closed subset A of X and each neighborhood $U \supset A$, there exists a neighborhood $V \supset A$ such that $\bar{V} \subset U$.

(iv) Subspaces and products of Hausdorff (or regular) spaces are Hausdorff (or regular).

(v) Examples.

(a) The space $X = \mathbb{R}_K$ is a Hausdorff space, as it is finer than $\mathbb{R}$. However, X is not regular, since the pair of subsets $\{\{0\}, K\}$ can never be separated by open sets.

(b) The space $X = \mathbb{R}_\ell$ is Hausdorff (and hence T_1), as it's finer than $\mathbb{R}$. Furthermore, X is normal, for consider any pair $\{A, B\}$ of disjoint closed subsets of X. Then for each $a \in A$, there exists a basic open set $[a, x_a) \subset X \setminus B$, and for each $b \in B$, there exists a basic open set $[a, x_b) \subset X \setminus A$. Then the open sets

$$U = \bigcup_{a \in A} [a, x_a) \text{ and } V = \bigcup_{b \in B} [b, x_b)$$

separate $\{A, B\}$.

(vi) A regular second countable space is normal.

(vii) A metrizable space is normal.

(viii) A compact Hausdorff space is normal.

(ix) An arbitrary product of normal spaces is not necessarily normal. For example, $\mathbb{R}$ is normal, but $\mathbb{R}^J$ is not normal in the product topology.

(x) (Tietze Extension Theorem) Let X be a normal space, and A be a closed subset of X. Then:

  (a) Any continuous map $A \to [a, b] \subset \mathbb{R}$ can be extended to a continuous map $X \to [a, b]$.

  (b) Any continuous map $A \to \mathbb{R}$ can be extended to a continuous map $X \to \mathbb{R}$.

(xi) (Urysohn Lemma) Let X be a normal space, and A, B be disjoint closed subsets of X. Then there exists a continuous map

$$f : X \to [a, b] \subset \mathbb{R}$$

such that $f(A) = \{a\}$ and $f(B) = \{b\}$.

(xii) Every regular space second countable space can be imbedded in the space $\mathbb{R}^\infty$ with the product topology.

(xiii) (Urysohn Metrization Theorem) Every regular second countable space is metrizable.

1.15 Imbeddings of manifolds

(i) An *m-manifold* is a Hausdorff second countable space X in which each $x \in X$ has a neighborhood $U_x \ni x$ such that $U_x \approx \mathbb{R}^m$.

(ii) Examples of manifolds.

  (a) A 1-manifold is called a *curve*. Examples of 1-manifolds are $\mathbb{R}$, S^1 etc.

(b) A 2-manifold is also called a *surface*. The 2-sphere S^2 and the torus $S^1 \times S^1$ are orientable (two-sided) 2-manifolds, while the Möbius band and the Klein bottle are non-orientable (one-sided) 2-manifolds.

(c) Examples of 3-manifolds are S^3, $S^2 \times S^1$, $D^2 \times S^1$, $S^1 \times S^1 \times S^1$ etc.

(iii) Let X be a topological space. The *support* of a function $\phi : X \to \mathbb{R}$ (denoted by $\mathrm{Supp}(\phi)$) is defined by

$$\mathrm{Supp}(\phi) = \overline{\phi^{-1}(\mathbb{R} \setminus \{0\})}.$$

(iv) Let $\{U_1, \ldots, U_n\}$ be a finite open covering for a space X. The an indexed family of continuous functions

$$\phi_i : X \to [0,1], \text{ for } 1 \leq i \leq n,$$

is said to be the *partition of unity dominated by* $\{U_i\}_{i=1}^n$ if:

(a) $\mathrm{Supp}(\phi_i) \subset U_i$, for $1 \leq i \leq n$, and

(b) $\sum_{i=1}^{n} \phi_i(x) = 1$, for each $x \in X$.

(v) (Existence of partitions of unity). Let $\{U_i\}_{i=1}^n$ be a finite open covering for a normal space X. Then there exists a partition of unity dominated by $\{U_i\}_{i=1}^n$.

(vi) Let X be a compact m-manifold. Then X can be imbedded in $\mathbb{R}^n$, for some suitably large positive integer n.